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Time-dependent critical layers in shear flows on the beta-plane

By FRED J. HICKERNELL

Department of Mathematics, University of Southern California, University Park, Los Angeles

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The problem of a finite-amplitude free disturbance of an inviscid shear flow on the beta-plane is studied. Perturbation theory and matched asymptotics are used to derive an evolution equation for the amplitude of a singular neutral mode of the Kuo equation. The effects of time-dependence, nonlinearity and viscosity are included in the analysis of the critical-layer flow. Nonlinear effects inside the critical layer rather than outside the critical layer determine the evolution of the disturbance. The nonlinear term in the evolution equation is some type of convolution integral rather than a simple polynomial. This makes the evolution equation significantly different from those commonly encountered in fluid wave and stability problems.

1. Introduction

Asymptotic methods have been used successfully to derive approximate solutions of the equations of fluid motion. The problem of small two-dimensional disturbances of an inviscid incompressible basic flow $U(y)\,\mathbf{i}$ has been studied for some time. Rayleigh (1880) considered infinitesimal disturbances and presented the seminal ideas for the linear stability theory of a homogeneous flow. The extensive results which followed have been reviewed by Drazin & Howard (1966). More recently, perturbation series and multiple scales have been employed to study finite-amplitude disturbances.

A major difficulty in this asymptotic analysis is the treatment of the flow in the critical layers, places where the basic-state velocity $U(y)$ equals the phase velocity c of the disturbance. The linearized equations of fluid motion can be solved by taking Fourier transforms in x and t. The dependence of a Fourier mode on y is given by an ordinary differential-equation boundary-value problem, which also determines the dispersion relation. This differential equation is in general singular at places where $U-c$ vanishes. The singularity in the equation produces a singularity in the velocity of the disturbance. The singularity worsens for higher-order terms in a perturbation series.

To describe a finite-amplitude disturbance in a shear flow with critical layers, the equations of motion must be solved separately inside and outside the critical layers, and then the solutions must be matched. As time evolves the effects of small viscosity and nonlinearity become important earlier inside the critical layers than they do outside. Schade (1964); Huerre (1980) and Burns & Maslowe (1983) derived amplitude-evolution equations for small disturbances in inviscid flows where the effect of viscosity smooths out the flow in the critical layers. Benney & Bergeron (1969); Davis (1969); Kelly & Maslowe (1970); Maslowe (1972) and Haberman (1972) considered how nonlinear effects, rather than, or in addition to viscous effects, might determine the flow in the critical layers. They calculated small, steady disturbances with strongly nonlinear critical-layer flows. The idea of including nonlinearity in the critical-layer analysis was later used in deriving evolution equations for small disturbances. Benney & Maslowe (1975) and Huerre & Scott (1980) studied homogeneous shear flows, and Brown & Stewartson (1978*b*) studied a stratified shear flow. Redekopp (1977); Maslowe & Redekopp (1979, 1980) and Stewartson (1981) studied disturbances with large horizontal scale in flows with critical layers. Stewartson (1978, 1981), Brown & Stewartson (1978*a*, 1980, 1982*a*, *b*) and Warn & Warn (1978) studied forced disturbances in flows with critical layers.

The purpose of the present study is to understand better how the critical-layer flow evolves when the effects of nonlinearity, viscosity and time dependence are all of comparable importance and how this in turn effects the evolution of the disturbance outside the critical layers. The particular problem chosen is that of a finite-amplitude disturbance of a marginally stable shear flow in the beta-plane model of Rossby *et al.* (1939). Outside the critical layers the disturbance is assumed to be, to lowest order, a singular neutral mode of the linearized equations of motion.

There are at least three distinguishing features of this analysis. First, the flow inside the critical layers is not assumed to be steady or quasi-steady; the vorticity equation there depends explicitly on time. Secondly, the flow in the critical layers is weakly nonlinear, just as is the case for the outer flow. Each term in the perturbation series for the critical-layer flow is governed by a linear equation. The nonlinear terms enter as non-homogeneities. Thirdly, the nonlinear term in the evolution equation for the disturbance comes only from nonlinear interactions inside the critical layers. The nonlinear interactions outside the critical layers are not as strong.

2. Formulation

The flow under consideration is incompressible, homogeneous, two-dimensional and unbounded. It is convenient to use a stream function Ψ to represent the velocity field. The vorticity equation on the beta-plane is

$$\left[\frac{\partial}{\partial t}+\Psi_y\frac{\partial}{\partial x}-\Psi_x\frac{\partial}{\partial y}\right]\nabla^2\Psi+\beta\Psi_x=\nu\nabla^2\nabla^2\Psi,$$

where x represents longitude and y represents latitude. The parameter β is the constant approximation to the y-derivative of the Coriolis parameter, and ν is the kinematic viscosity. All of the above variables and parameters have been made dimensionless with respect to the magnitude of the velocity shear and the length of the shear layer in the basic state flow.

The basic-state velocity is $U_{\mathrm{B}}(y,t)\,\boldsymbol{i}$. It depends on time because of the slow viscous dissipation. On the timescales considered in this problem the basic flow can be accurately represented by a power series in νt:

$$U_{\mathrm{B}yt}=\nu U_{\mathrm{B}yyy},$$

$$U_{\mathrm{B}}(y,t)=U(y)+\nu t U''(y)+\dots.$$

$U(y)$ is an arbitrary function with the property that $U\rightarrow U_{\pm\infty}$ and $U''\rightarrow 0$ as $y\rightarrow\pm\infty$. Only the first term in the above series contributes to the amplitude-evolution equation for the disturbance. An alternative to having a slowly changing basic flow is to assume an artificial body force that cancels the effect of viscous dissipation and preserves a steady basic state.

A small two-dimensional disturbance of magnitude ϵ is now added to the basic flow. It is presumed to propagate in the x-direction with a real phase speed c

determined by linear-stability theory. In addition to its dependence on $x-ct$ and y, the disturbance also depends on a slow time variable

$$T \equiv \mu t, \quad \mu \ll 1. \tag{2.1}$$

Following standard perturbation methods the stream function ψ of the disturbance is expanded in powers of ϵ. The requirement that no secularities be present in the expansion determines the evolution of the amplitude of the disturbance in terms of T. In terms of this new variable the vorticity equation becomes

$$\left[\mu\frac{\partial}{\partial T}+(U-c+\psi_y)\frac{\partial}{\partial x}-\psi_x\frac{\partial}{\partial y}\right]\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\psi-(U''-\beta)\psi_x = \nu\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]^2\psi+O(\nu t\psi). \tag{2.2}$$

A Galilean transformation has been employed to replace $x-ct$ by x.

The proper relationship between ϵ and μ is determined by balancing the effect of slow time dependence and the effect of small nonlinearity. The disturbance is assumed to be uniformly small initially, and then the magnitudes of the effects of time-dependence and nonlinearity are calculated. These effects create secular terms in the perturbation expansion for the stream function. The balance is achieved by choosing the earliest time (largest μ) for which these secularities are the same size. The evolution equation then arises from the condition that the sum of the secularities vanishes.

The relationship between ϵ and μ found for this problem is not one used previously, and a derivation is now given. Outside the critical layers the magnitude of ψ and its derivatives is $O(\epsilon)$. The advective nonlinearities in the vorticity equation produce terms in the series for ψ with magnitudes $O(\epsilon^2)$, $O(\epsilon^3)$, etc. The largest of these which creates a secularity in the perturbation expansion is the $O(\epsilon^3)$ term. This explains the cubic nonlinearities which often arise in evolution equations for disturbances with finite wavenumber. Inside the critical layers, the magnitudes of ψ and its derivatives are different. The dominant terms in (2.2) are $\mu\psi_{yyT}$, $(U-c)\psi_{yyx}$, $(U''-\beta)\psi_x$, and possibly the viscous term. Balancing the $\mu\psi_{yyT}$ and $(U-c)\psi_{yyx}$ terms determines the thickness of the critical layer to be $O(\mu)$. Initially, the disturbance vorticity is $O(\epsilon)$, but the balance between $\mu\psi_{yyT}$ and $(U''-\beta)\psi_x$ causes it to become $O(\epsilon\mu^{-1})$ for T of order unity. Because the critical layer has thickness μ, $\psi_{yyy} = O(\epsilon\mu^{-2})$. Balancing $\mu\psi_{yyT}$ with the advective nonlinearity $\psi_x\psi_{yyy}$ gives an $O(\epsilon^2\mu^{-3})$ contribution to ψ_{yy}. This quadratic nonlinearity itself does not create a secularity in the expansion for ψ. As is often the case, it is necessary to proceed to cubic nonlinearities. The interaction of $\psi_x = O(\epsilon)$ and $\psi_{yyy} = O(\epsilon^2\mu^{-4})$ makes an $O(\epsilon^3\mu^{-5})$ contribution to ψ_{yy}. When this is integrated across a critical layer, an $O(\epsilon^3\mu^{-4})$ jump in the velocity ψ_y is produced. This effect on the outer flow is larger, and therefore more important, than the $O(\epsilon^3)$ effect mentioned above. Balancing the $O(\epsilon^3\mu^{-4})$ nonlinear effect with the $O(\epsilon\mu)$ effect of slow time yields the following relation between ϵ and μ:

$$\mu = \epsilon^{\frac{2}{5}}.$$

Viscosity is also scaled in terms of ϵ:

$$\nu = \lambda\mu^3 = \lambda\epsilon^{\frac{6}{5}}.$$

The parameter λ is order unity and is similar, but not identical with, that introduced by Benney & Bergeron (1969). The relationship between ν and μ is what one would normally expect in a viscous layer. This is not, however, a viscous critical layer in the sense that viscous effects dominate the flow. The parameter λ can be set identically to zero without changing the following analysis or resulting evolution equation in any essential way. Viscosity is scaled in the above manner because if it is smaller, then it has no effect on the evolution of the disturbance over the time considered, and, if it is larger, then its effect dominates the critical-layer flow.

The parameter β is scaled in the following way:

$$\beta = \beta_0 + \beta_1 \mu = \beta_0 + \beta_1 \epsilon^{\frac{2}{5}}.$$

The paramater β_0 is some basic value, and the parameter β_1 is a measure of the deviation of β from it. For the following analysis and resulting evolution equation to make the most sense, β_0 should be the value that makes the flow marginally stable.

3. Outer flow

Outside the critical layers the stream function of the disturbance is written as the following perturbation series:

$$\psi = \epsilon\psi_1 + \epsilon^{\frac{6}{5}}\psi_2 + \epsilon^{\frac{7}{5}}\psi_3 + \dots .$$

The first term in the series is a mode of the linearized version of the vorticity equation (2.2) with longitudinal wavenumber α. The second term in the series is a combination of modes with wavenumbers 0 and 2α which are forced by the flow in the critical layers. Although there is a component of the critical-layer flow with the proper magnitude and wavenumber to force these modes, this component produces no velocity jump across the critical layers. So, in fact, ψ_2 is zero. The component of the third term in the series with wavenumber α obeys a non-homogeneous equation. The solvability condition for this equation leads to the evolution equation for the amplitude of the disturbance.

3.1. *The mode of the linearized equation*

The first term in the series for ψ has the form

$$\psi_1 = A(T)\,\phi(y)\,\mathrm{e}^{\mathrm{i}\alpha x} + *,$$

where α is assumed to be positive throughout this paper. The latitudinal structure $\phi(y)$ satisfies the following equation derived by Kuo (1949):

$$\phi'' - \left[\alpha^2 + \frac{U'' - \beta_0}{U - c}\right]\phi = 0. \tag{3.1}$$

There are boundary conditions on $\phi(y)$ as y approaches infinity. (The choice to consider an unbounded rather than a bounded flow does not effect the analysis in a major way.) As y approaches infinity, $\phi(y)$ has the following asymptotic behaviour:

$$\phi(y) \sim \phi_{\pm\infty}\,\mathrm{e}^{\mathrm{i}k_{\pm}y} \quad \text{as} \quad y \to \pm\infty, \tag{3.2}$$

$$k_{\pm}^2 = \frac{\beta_0}{U_{\pm\infty} - c} - \alpha^2. \tag{3.3a}$$

The latitudinal wavenumber $k_{\pm}$ may be either imaginary or real, corresponding to a decaying or radiating mode respectively. If $k_{\pm}$ is imaginary then $\mathrm{i}k_{\pm}$ must have the sign that guarantees a decaying solution as $y \to \pm\infty$. If $k_{\pm}$ is real then its sign is determined by the radiation boundary condition. The group velocity in the latitudinal direction should have the sign $\pm$ for $y \to \pm\infty$. In this problem the group velocity is

$$c_g = \frac{\partial(\alpha c)}{\partial k} = \frac{2\alpha k_\pm \beta_0}{\alpha^2 + k_\pm^2}.$$

So the sign of $k_\pm$ is chosen as follows:

$$\mathrm{i}k_\pm \lesseqgtr 0 \quad \text{for} \quad k_\pm^2 < 0, \tag{3.3b}$$

$$k_\pm \beta_0 \gtrless 0 \quad \text{for} \quad k_\pm^2 > 0. \tag{3.3c}$$

The solution of the Kuo equation (3.1) subject to the boundary conditions (3.2) and (3.3) determines c in terms of α. One additional piece of information is needed, however. That is the jump conditions on the eigenfunction across the critical layers. Near a critical layer $y = y_c$ the eigenfunction ϕ has the following asymptotic behaviour:

$$\begin{aligned}\phi(y) \sim{}& a_\pm \left\{ y - y_c + \frac{U_c'' - \beta_0}{2U_c'}(y-y_c)^2 + \dots \right\} \\ &+ b_\pm \left\{ 1 + \left[\frac{U_c'(U_c''' + \alpha^2) - (2U_c'' - \tfrac{3}{2}\beta_0)(U_c'' - \beta_0)}{2U_c'^2} \right](y-y_c)^2 + \dots \right. \\ &\left. + \frac{U_c'' - \beta_0}{U_c'} \left[y - y_c + \frac{U_c'' - \beta_0}{2U_c'}(y-y_c)^2 + \dots \right] \ln|y-y_c| \right\} \quad \text{as} \quad y \to y_c^\pm,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\phi'(y) \sim{}& a_\pm \left\{ 1 + \frac{U_c'' - \beta_0}{U_c'}(y-y_c) + \dots \right\} \\ &+ b_\pm \left\{ \left[\frac{U_c'(U_c''' + \alpha^2) - (2U_c'' - \tfrac{3}{2}\beta_0)(U_c'' - \beta_0)}{U_c'^2} \right](y-y_c) + \dots \right. \\ &+ \frac{U_c'' - \beta_0}{U_c'} \left[1 + \frac{U_c'' - \beta_0}{U_c'}(y-y_c) + \dots \right] \ln|y-y_c| \\ &\left. + \frac{U_c'' - \beta_0}{U_c'} \left[1 + \frac{U_c'' - \beta_0}{2U_c'}(y-y_c) + \dots \right] \right\} \quad \text{as} \quad y \to y_c^\pm,\end{aligned} \tag{3.5}$$

$$U_c' \equiv U'(y_c), \quad U_c'' \equiv U_c''(y_c), \quad U_c''' \equiv U'''(y_c), \quad \text{etc.}$$

The jumps $a_+ - a_-$ and $b_+ - b_-$ are found by analysing the flow in the critical layer.

There are few known examples of singular neutral modes of the Kuo equation ($U'' - \beta_0 \neq 0$ at some place where $U - c = 0$). There are no examples known to the author where the neutral mode of a marginally stable flow is singular. This is no reason, however, to doubt their existence. Most of those velocity profiles for which the Kuo equation has been solved are symmetric or antisymmetric. Because of these symmetries, the neutral modes tend to be regular ($U'' - \beta_0 = 0$, where $U - c = 0$). Suppose, on the other hand, that a velocity profile lacks symmetry. To be specific, suppose that, for each c in the range of possible values of U, $U - c = 0$ at more than one place, and U''does not have the same value at these places. All neutral modes must then be singular. Furthermore, sufficiently large β_0 always stabilizes a shear flow with bounded U''. The neutral mode that is present when the flow is marginally stable is the one of interest. Unfortunately, numerical calculations may be the only way to find it.

Presently, an example is presented to show that singular neutral modes do exist.

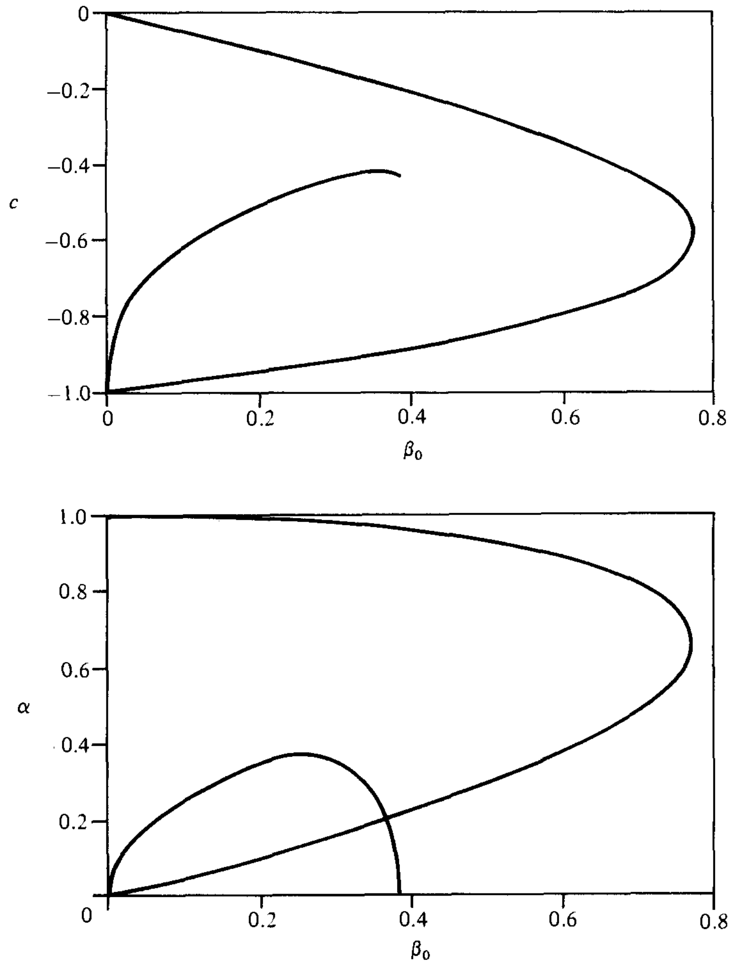

FIGURE 1. The eigenvalues for the neutral modes of the Kuo equation with $U(y) = \tanh y$.

The weakness of this example, however, is that the singular neutral modes disappear well before the flow stabilizes. The velocity profile $U(y) = \tanh y$ has regular neutral modes which were discovered by Howard & Drazin (1964):

$$\alpha^2 = 1-c^2, \quad \beta_0 = -2c(1-c^2), \quad -1 \leqslant c \leqslant 1,$$
$$\phi = (1-\tanh y)^{\frac{1}{2}(1+c)}(1+\tanh y)^{\frac{1}{2}(1-c)}.$$

The eigenfunctions for these modes decay as y approaches infinity in either direction. A radiating neutral mode was suggested by the calculations of Dickinson & Clare (1973) and calculated by the author. This mode is singular when it is neutral. Figures 1–5 show some neutral and unstable eigenvalues for positive values of β_0. The radiating neutral mode radiates as $y \to +\infty$ and decays as $y \to -\infty$. As β_0 increases, the flow is stabilized. When β_0 reaches about 0.38, the radiating neutral mode and the unstable modes contiguous to it disappear. The flow does not become linearly stable, however, until $\beta_0 = 4 \times 3^{-\frac{3}{2}}$. Because of the antisymmetry of $U(y)$ the eigenvalues c for negative values of β_0 are minus the eigenvalues for positive β_0.

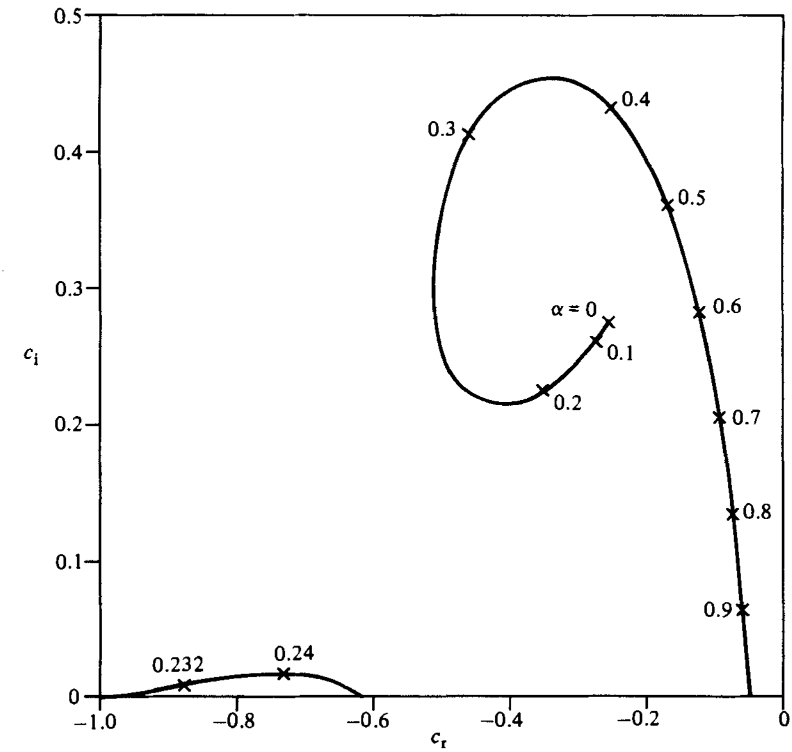

FIGURE 2. The complex wave speeds for modes of the Kuo equation with $U(y) = \tanh y$ and $\beta_0 = 0.1$.

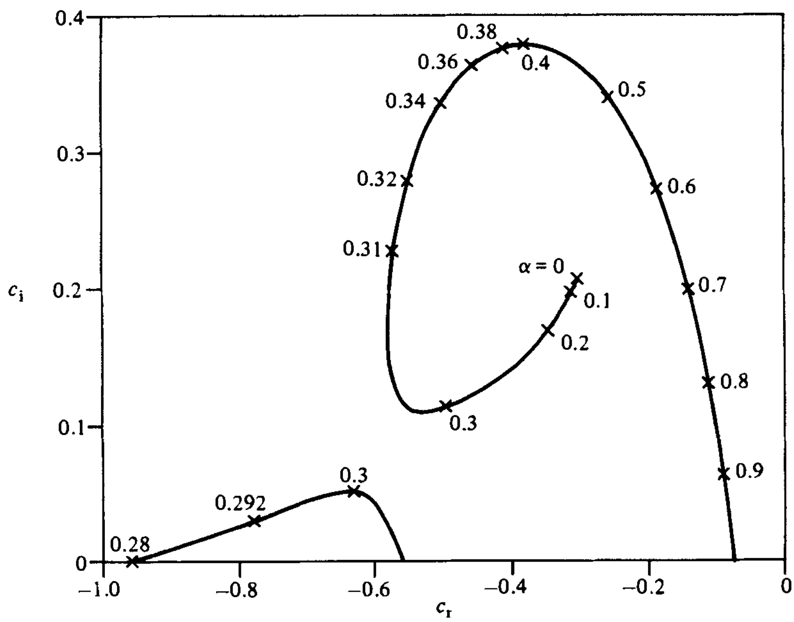

FIGURE 3. The complex wave speeds for modes of the Kuo equation with $U(y) = \tanh y$ and $\beta_0 = 0.15$.

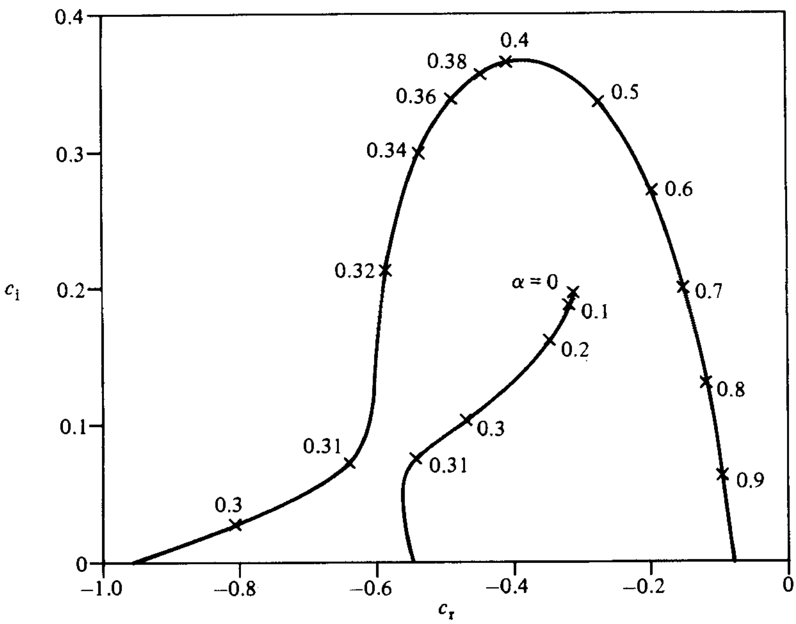

FIGURE 4. The complex wave speeds for modes of the Kuo equation with $U(y) = \tanh y$ and $\beta_0 = 0.16$.

3.2. *The* $O(\epsilon^{\frac{6}{5}})$ *term*

The solution of the vorticity equation for ψ_2 is similar to the solution for ψ_1:

$$\psi_2 = B(T)\,\phi_2^{(1)}(y) + C(T)\,\phi_2^{(2)}(y)\,\mathrm{e}^{2\mathrm{i}\alpha x} + *.$$

The functions $\phi_2^{(1)}$ and $\phi_2^{(2)}$ satisfy the same equation and boundary conditions as ϕ except that the wavenumber is no longer α but 0 or 2α. The jump conditions across the critical layers may be different, however. The jumps may arise from the nonlinear interactions of the fundamental mode in the critical layers, and so the functions $\phi_2^{(1)}$ and $\phi_2^{(2)}$ need not be eigenfunctions of the Kuo equation.

3.3. *The* $O(\epsilon^{\frac{7}{5}})$ *term*

The part of ψ_3 with wavenumber α takes the form

$$\psi_3^{(1)} = \phi_3(y, T)\,\mathrm{e}^{\mathrm{i}\alpha x} + *,$$

and ϕ_3 satisfies the following non-homogeneous equation:

$$\mathrm{i}\alpha\left\{\phi_{3yy} - \left[\alpha^2 + \frac{U'' - \beta_0}{U - c}\right]\phi_3\right\} = \left\{-\frac{U'' - \beta_0}{(U-c)^2}A_T - \frac{\mathrm{i}\alpha\beta_1}{U-c}A\right\}\phi. \tag{3.6}$$

As y approaches infinity, $\mathrm{i}\alpha\phi_3$ has the following asymptotic behaviour:

$$\mathrm{i}\alpha\phi_3 \sim \mathrm{i}\alpha\phi_{3\pm\infty}(T)\,\mathrm{e}^{\mathrm{i}k_\pm y} + \frac{1}{2\mathrm{i}k_\pm}\left\{\frac{\beta_0}{(U_{\pm\infty} - c)^2}A_T - \frac{\mathrm{i}\alpha\beta_1}{U_{\pm\infty} - c}A\right\}\phi_{\pm\infty}\,y\,\mathrm{e}^{\mathrm{i}k_\pm y} \quad \text{as} \quad y \to \pm\infty.$$

The first term comes from the solution of the homogeneous equation, and $k_\pm$ is given by (3.3). The second term comes from the solution of the non-homogeneous equation.

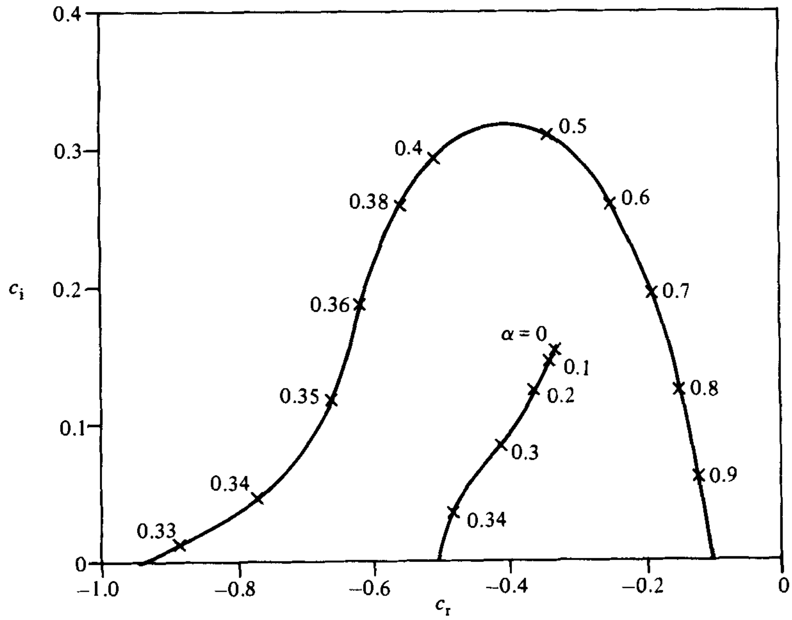

FIGURE 5. The complex wave speeds for modes of the Kuo equation with $U(y) = \tanh y$ and $\beta_0 = 0.2$.

It may become large as y approaches infinity, but this secularity can be removed by introducing a 'long' latitudinal variable.

Near a critical layer $i\alpha\phi_3$ has the following asymptotic behaviour:

$$
\begin{aligned}
i\alpha\phi_3(y,T) \sim \ln|y-y_c| \Big\{ & \frac{U''-\beta_0}{U'^2_c} b_\pm A_T \\
& + \left[\frac{U''_c(U''_c-\beta_0)-U'_c U''_c}{U'^3_c} b_\pm - \frac{U''_c-\beta_0}{U'^2_c} a_\pm \right] A_T (y-y_c) \\
& - \frac{i\alpha\beta_1}{U'_c} b_\pm A (y-y_c) + \frac{U''_c-\beta_0}{U'_c} b_{3\pm}(T)\,(y-y_c) + \dots \Big\} \\
& + b_{3\pm}(T) + a_{3\pm}(T)\,(y-y_c) + \dots \quad \text{as} \quad y \to y_c^\pm, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
i\alpha\phi_{3y}(y,T) \sim \ln|y-y_c| \Big\{ & -\frac{i\alpha\beta_1}{U'_c} b_\pm A + \frac{U''_c-\beta_0}{U'_c} b_{3\pm}(T) \\
& + \left[\frac{U''_c(U''_c-\beta_0)-U'_c U'''_c}{U'^3_c} b_\pm - \frac{U''_c-\beta_0}{U'^2_c} a_\pm \right] A_T + \dots \Big\} \\
& + \Big\{ \frac{U''_c-\beta_0}{U'^2_c} b_\pm A_T \frac{1}{y-y_c} - \frac{i\alpha\beta_1}{U'_c} b_\pm A + \frac{U''_c-\beta_0}{U'_c} b_{3\pm}(T) \\
& + \left[\frac{U''_c(U''_c-\beta_0)-U'_c U'''_c}{U'^3_c} b_\pm - \frac{U''_c-\beta_0}{U'^2_c} a_\pm \right] A_T + a_{3\pm}(T) + \dots \Big\} \\
& \text{as} \quad y \to y_c^\pm. \qquad (3.8)
\end{aligned}
$$

A solvability condition for the above boundary-value problem (3.6) is derived by multiplying both sides of the equation by ϕ and integrating over all y, excluding the critical layers. After integrating by parts the solvability condition becomes the following:

$$-A_T \lim_{R\to+\infty}\left[\fint_{-R}^{R}\frac{U''-\beta_0}{(U-c)^2}\phi^2\,\mathrm{d}y-\frac{\beta_0\phi_{-\infty}^2\,\mathrm{e}^{-2\mathrm{i}k_-R}}{2\mathrm{i}k_-(U_{-\infty}-c)^2}+\frac{\beta_0\phi_{+\infty}^2\,\mathrm{e}^{2\mathrm{i}k_+R}}{2\mathrm{i}k_+(U_{+\infty}-c)^2}\right]$$

$$-\mathrm{i}\alpha\beta_1 A \lim_{R\to+\infty}\left[\fint_{-R}^{R}\frac{\phi^2}{U-c}\mathrm{d}y+\frac{\phi_{-\infty}^2\,\mathrm{e}^{-2\mathrm{i}k_-R}}{2\mathrm{i}k_-(U_{-\infty}-c)}-\frac{\phi_{+\infty}^2\,\mathrm{e}^{2\mathrm{i}k_+R}}{2\mathrm{i}k_+(U_{+\infty}-c)}\right]$$

$$+\mathrm{i}\alpha\sum_{y_c}\left[\phi_{3y}\phi-\phi_3\phi'\right]_{y_c^-}^{y_c^+}=0. \qquad (3.9)$$

The notation $\fint$ means integration excluding the critical layers. The quantity $\phi_{3y}\phi-\phi_3\phi'$ arises from the integration by parts of the left-hand side of (3.6). Its value at $y=\pm R$ contributes the terms with factors of $\mathrm{e}^{\pm 2\mathrm{i}k_\pm R}$. Its jump across the critical layers contributes the last term in the above equation. This jump is determined in §4.

4. Critical-layer flow

Inside a critical layer, $y-y_c \ll 1$, the stream function of the disturbance is written as

$$\hat\psi=\epsilon\hat\psi_1+\epsilon^{\frac{6}{5}}\hat\psi_2+\epsilon^{\frac{7}{5}}\hat\psi_3+\epsilon^{\frac{8}{5}}\hat\psi_4+\epsilon^{\frac{9}{5}}\hat\psi_5+\dots,$$

where ^ distinguishes this series from that for the outer flow. Terms of size $O(\epsilon^p(\ln\mu)^q)$ are included in the terms of size $O(\epsilon^p)$. Since $\hat\psi$ is expected to vary on the latitudinal lengthscale of μ^{-1}, a new variable

$$Y\equiv\epsilon^{-\frac{2}{5}}(y-y_c)$$

is introduced to replace y. The basic flow is expanded in a Taylor series in terms of Y:

$$U-c\sim\epsilon^{\frac{2}{5}}U_c'\,Y+\epsilon^{\frac{4}{5}}\tfrac{1}{2}U_c''\,Y^2+\dots,$$

$$U''-\beta\sim U_c''-\beta_0+\epsilon^{\frac{2}{5}}(U_c'''Y-\beta_1)+\dots.$$

4.1. *The $O(\epsilon)$ and $O(\epsilon^{\frac{6}{5}})$ terms*

Substituting the above series for the stream function into the vorticity equation (2.2) yields the following equation for $\hat\psi_1$:

$$\left.\begin{aligned}\hat\psi_{1YYT}+U_c'\,Y\hat\psi_{1YYx}-\lambda\hat\psi_{1YYYY}&=0,\\ \hat\psi_{1YY}(x,Y,0)&=0.\end{aligned}\right\} \qquad (4.1)$$

The initial condition comes from the assumption that $\hat\psi_{yy}=O(\epsilon)$ initially. Thus $\hat\psi_{YY}=O(\epsilon^{\frac{9}{5}})$ initially. The first non-trivial initial value occurs for $\hat\psi_5$. The solution of (4.1) that matches the outer flow is

$$\hat\psi_1=\hat A(T)\,\mathrm{e}^{\mathrm{i}\alpha x}+*.$$

This relates the constants $b_\pm$ in (3.4) and (3.5):

$$b_+=b_-=b,\quad \hat A(T)=bA(T). \qquad (4.2)$$

The equation and initial condition for $\hat\psi_2$ is similar to that for $\hat\psi_1$. The solution that matches the outer flow is

$$\hat\psi_2=\hat B(T)+\hat C(T)\,\mathrm{e}^{2\mathrm{i}\alpha x}+*.$$

4.2. *The* $O(\epsilon^{\frac{3}{2}})$ *term*

The equation for $\hat{\psi}_3$ is the first non-homogeneous equation for the flow in the critical layers, and its solution is non-trivial:

$$\hat{\psi}_{3YYT} + U_c' Y \hat{\psi}_{3YYx} - \lambda \hat{\psi}_{3YYYY} = (U_c'' - \beta_0)\hat{\psi}_{1x} = \mathrm{i}\alpha(U_c'' - \beta_0)\hat{A}(T)\,\mathrm{e}^{\mathrm{i}\alpha x} + *, \quad (4.3a)$$

$$\hat{\psi}_{3YY}(x, Y, 0) = 0. \quad (4.3b)$$

Focusing on the part of $\hat{\psi}_3$ with wavenumber α simplifies this equation:

$$\hat{\psi}_3^{(1)} = \hat{\phi}_3(Y, T)\,\mathrm{e}^{\mathrm{i}\alpha x} + *,$$

$$\hat{\phi}_{3YYT} + \mathrm{i}\alpha U_c' Y \hat{\phi}_{3YY} - \lambda \hat{\phi}_{3YYYY} = \mathrm{i}\alpha(U_c'' - \beta_0)\hat{A}(T).$$

Taking a Fourier transform in Y reduces this equation to a first-order partial differential equation in the latitudinal wavenumber K and T. Switching to the variables K and $\tilde{T} \equiv T + K/\alpha U_c'$ reduces this latter equation to a first-order ordinary differential equation in K which is easily solved:

$$Z_3(K, T) = \mathscr{F}\{\hat{\phi}_{3YY}(\cdot, T)\}(K) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\phi}_{3YY}(Y, T)\,\mathrm{e}^{-\mathrm{i}KY}\,\mathrm{d}Y,$$

$$\hat{\phi}_{3YY}(Y, T) = \mathscr{F}^{-1}\{Z_3(\cdot, T)\}(Y) = \int_{-\infty}^{+\infty} Z_3(K, T)\,\mathrm{e}^{\mathrm{i}KY}\,\mathrm{d}K,$$

$$Z_{3T} - \alpha U_c' Z_{3K} + \lambda K^2 Z_3 = \mathrm{i}\alpha(U_c'' - \beta_0)\hat{A}(T)\,\delta(K),$$

$$Z_3 = \frac{-\mathrm{i}(U_c'' - \beta_0)}{|U_c'|}\hat{A}\left(T + \frac{K}{\alpha U_c'}\right)\mathrm{e}^{\lambda K^3/3\alpha U_c'}$$
$$\times \{H(-U_c')[H(K + \alpha U_c' T) - H(K)] + H(U_c')[H(-K - \alpha U_c' T) - H(-K)]\}.$$

Because the solutions of the vorticity equation in the critical layers do not decay as $Y \to \pm\infty$, their Fourier transforms involve distributions such as the Dirac delta function $\delta(K)$. In addition to this distribution, other distributions and their Fourier transforms are used in the analysis that follows. $\delta'(K)$ is the derivative of $\delta(K)$, and $H(K)$ is the Heaviside step function, which is defined as follows:

$$H(K) = \begin{cases} 0 & (K < 0), \\ 1 & (0 < K). \end{cases}$$

The distributions pf $(H(\pm K)\,K^{-n})$ are defined in terms of the finite parts of singular integrals:

$$\mathrm{pf}\int_{-\infty}^{+\infty} H(\pm K)\,K^{-n} f(K)\,\mathrm{d}K = \int_{-\infty}^{+\infty} \mathrm{pf}\,(H(\pm K)\,K^{-n}) f(K)\,\mathrm{d}K = \lim_{\sigma \downarrow 0}\left\{\int_{\pm\sigma}^{\pm\infty} K^{-n} f(K)\,\mathrm{d}K\right.$$
$$\left. + f(0)\frac{(\pm\sigma)^{-n+1}}{-n+1} + \dots + \frac{f^{(n-2)}(0)}{(n-2)!}(\pm\sigma) + \frac{f^{(n-1)}(0)}{(n-1)!}\ln\sigma\right\}$$

for any smooth test function $f(K)$. The notation pf denotes the finite part of an integral. The finite part is found by expanding the integrand in a series about the singularity, determining the singular part, and then subtracting this part off. Stakgold (1979) discusses distributions and their Fourier transforms in more detail.

The Fourier transform of $\hat{\phi}_{3Y}$ is found from Z_3:

$$\mathrm{i}K\mathscr{F}\{\hat{\phi}_{3Y}(\cdot,T)\}(K) = Z_3,$$

$$\begin{aligned}\mathscr{F}\{\hat{\phi}_{3Y}(\cdot,T)\}(K) = &-\frac{U_c''-\beta_0}{|U_c'|}\hat{A}\left(\frac{T+K}{\alpha U_c'}\right)\mathrm{e}^{\lambda K^3/3\alpha U_c'} \\ &+\left\{H(-U_c')\left[\frac{H(K+\alpha U_c' T)}{K}+\ln(-\alpha U_c' T)\,\delta(K)-\mathrm{pf}\left(\frac{H(K)}{K}\right)\right]\right. \\ &\left.+H(U_c')\left[\frac{H(-K-\alpha U_c' T)}{K}-\ln(\alpha U_c' T)\,\delta(K)-\mathrm{pf}\left(\frac{H(-K)}{K}\right)\right]\right\} \\ &+\hat{D}(T)\,\delta(K). \qquad (4.4)\end{aligned}$$

The singular distributions $\mathrm{pf}(H(\pm K)/K)$ defined above obey the following relations:

$$\mathrm{pf}\left(\frac{H(\pm K)}{K}\right) = \lim_{\sigma\downarrow 0}\left[\frac{H(\pm K-\sigma)}{K}\pm\ln(\sigma)\,\delta(K)\right]$$

$$\mathrm{pf}\left(\frac{H(\pm K)}{K}\right) = \mathscr{F}\{\mp[\ln|\cdot|+\gamma]+\tfrac{1}{2}\mathrm{i}\pi[H(\cdot)-H(-\cdot)]\}(K),$$

where γ is Euler's constant. This latter relation is used to invert the Fourier transform in (4.4). The expression for $\hat{\phi}_{3Y}$ in (4.4) is composed of a part which matches the outer solution for $Y\to\pm\infty$ and another part which oscillates rapidly like $\exp(\pm\mathrm{i}\alpha U_c' YT)$. This latter part presumably matches the transients from the initial-value problem in the outer flow which are not considered here. The asymptotic behaviour of $\hat{\phi}_{3Y}$ for large Y which matches the outer solution is the following:

$$\begin{aligned}\hat{\phi}_{3Y} \sim &\frac{U_c''-\beta_0}{U_c'}\hat{A}(T)[\ln|\epsilon^{\frac{2}{5}}Y|+\ln|\alpha U_c' T|+\gamma] \\ &+\frac{\mathrm{i}\pi(U_c''-\beta_0)}{2|U_c'|}\hat{A}(T)[H(Y)-H(-Y)]+\tilde{D}(T) \quad \text{as} \quad Y\to\pm\infty.\end{aligned}$$

Matching this with the outer solution (3.4) and (3.5) gives the following jump conditions for $a_\pm$:

$$a_+-a_- = \frac{\mathrm{i}\pi(U_c''-\beta_0)}{|U_c'|}b, \qquad (4.5)$$

$$\tilde{D}(T) = \left\{a_- - \frac{U_c''-\beta_0}{U_c'}[\ln|\alpha U_c' T|+\gamma-1]\,b+\frac{\mathrm{i}\pi(U_c''-\beta_0)}{2|U_c'|}b\right\}A(T). \qquad (4.6)$$

The Fourier transform of $\hat{\phi}_3$ and $\hat{\phi}_3$ itself are found from the Fourier transform of $\hat{\phi}_{3Y}$:

$$\mathrm{i}K\mathscr{F}\{\hat{\phi}_3(\cdot,T)\}(K) = \mathscr{F}\{\hat{\phi}_{3Y}(\cdot,T)\}(K),$$

$$\begin{aligned}\mathscr{F}\{\phi_3(\cdot,T)\}(K) = &\frac{\mathrm{i}(U_c''-\beta_0)}{|U_c'|}\hat{A}\left(T+\frac{K}{\alpha U_c'}\right)\mathrm{e}^{\lambda K^3/3\alpha U_c'} \\ &\times\left\{H(-U_c')\left[\frac{H(K+\alpha U_c' T)}{K^2}-\ln(-\alpha U_c' T)\,\delta'(K)+\frac{\delta(K)}{\alpha U_c' T}\right.\right. \\ &\left.-\mathrm{pf}\left(\frac{H(K)}{K^2}\right)\right]+H(U_c')\left[\frac{H(-K-\alpha U_c' T)}{K^2}+\ln(\alpha U_c' T)\,\delta'(K) \\ &\left.\left.-\frac{\delta(K)}{\alpha U_c' T}-\mathrm{pf}\left(\frac{H(-K)}{K^2}\right)\right]\right\}+\mathrm{i}\hat{D}(T)\,\delta'(K)+\hat{E}(T)\,\delta(K).\end{aligned}$$

The distributions $\mathrm{pf}\,(H(\pm K)/K^2)$ obey the following relations:

$$\mathrm{pf}\left(\frac{H(\pm K)}{K^2}\right) = \lim_{\sigma \downarrow 0}\left\{\frac{H(\pm K-\sigma)}{K^2} \mp \ln(\sigma)\,\delta'(K) - \frac{\delta(K)}{\sigma}\right\},$$

$$\mathrm{pf}\left(\frac{H(\pm K)}{K^2}\right) = \mathscr{F}\{\mp \mathrm{i}(\cdot)\,[\ln|\cdot| - 1 + \gamma] - \tfrac{1}{2}\pi(\cdot)\,[H(\cdot) - H(-\cdot)]\}\,(K).$$

The asymptotic expression for $\hat{\phi}_3$ determines $b_{3\pm}(T)$ of (3.7) and (3.8):

$$b_{3\pm}(T) = \frac{U''_c - \beta_0}{U'^2_c}\,b\left\{\frac{A}{T} + [\ln|\alpha U'_c T| + \gamma]\,\hat{A}_T\right\} \pm \frac{\mathrm{i}\pi(U''_c-\beta_0)}{2U'_c\,|U'_c|}\,bA_T + \mathrm{i}\alpha\tilde{E}(T). \quad (4.7)$$

4.3. *The* $O(\epsilon^{\frac{8}{5}})$ *term*

The equation for $\hat{\psi}_4$ contains the first instance of nonlinear interactions:

$$\begin{aligned}\hat{\psi}_{4YYT} + U'_c\,Y\hat{\psi}_{4YYx} - \lambda\hat{\psi}_{4YYYY} &= (U''_c - \beta_0)\,\hat{\psi}_{2x} + \hat{\psi}_{1x}\,\hat{\psi}_{3YYY} \\ &= 2\mathrm{i}\alpha(U''_c-\beta_0)\,\hat{C}(T)\,\mathrm{e}^{2\mathrm{i}\alpha x} + \mathrm{i}\alpha\hat{A}(T)\,\hat{\phi}_{3YYY}\,\mathrm{e}^{2\mathrm{i}\alpha x} - \mathrm{i}\alpha\hat{A}^*(T)\,\hat{\phi}_{3YYY} + *, \\ \hat{\psi}_{4YY}(x,Y,0) &= 0.\end{aligned}$$

The interesting part of this solution has wavenumbers 0 and 2α. The equation is again solved by taking a Fourier transform in Y. The first non-homogeneous term above is analogous to the non-homogeneous term in (4.3), and it produces an analogous velocity jump. The second two terms, which are forcing by the nonlinear interactions of the fundamental mode, produce no velocity jump. Assuming that modes of the Kuo equation do not exist for a wave speed of c and a wavenumber of either 0 or 2α, ψ_2 must vanish. This means that

$$\hat{\psi}_2 = 0.$$

4.4. *The* $O(\epsilon^{\frac{9}{5}})$ *term*

The solution of the equation for $\hat{\psi}_5$ gives the last piece of information needed to arrive at the evolution equation for A, the jump condition for $a_{3\pm}(T)$:

$$\begin{aligned}\hat{\psi}_{5YYT} + U'_c\,Y\hat{\psi}_{5YYx} - \lambda\hat{\psi}_{5YYYY} &= -\hat{\psi}_{1xxT} - U'_c\,Y\hat{\psi}_{1xxx} - \tfrac{1}{2}U''_c\,Y^2\hat{\psi}_{3YYx} \\ &\quad + (U''_c-\beta_0)\,\hat{\psi}_{3x} + (YU'''_c - \beta_1)\,\hat{\psi}_{1x} + \hat{\psi}_{1x}\,\hat{\psi}_{4YYY}, \\ \hat{\psi}_{5YY}(x,Y,0) &= F(x).\end{aligned}$$

The initial condition is independent of Y because the initial critical-layer flow is assumed not to have a rapid latitudinal variation. This equation is solved in the same manner as before, and the velocity jump is calculated. From this the jump in $a_{3\pm}(T)$ is found:

$$\begin{aligned}a_{3+}(T) - a_{3-}(T) &= \frac{\pi\alpha\beta_1}{|U'_c|}\,bA + \frac{\mathrm{i}\pi(U''_c-\beta_0)}{|U'_c|}\left\{\mathrm{i}\alpha\tilde{E} - \frac{\tilde{D}_T}{U'_c}\right\} \\ &\quad + \frac{\mathrm{i}\pi[(U''_c-\beta_0)\,(2U''_c-\beta_0) - U'_c\,U'''_c]}{U'^2_c\,|U'_c|}\,bA_T \\ &\quad + \pi\alpha^5 U'_c(U''_c-\beta_0)\,b\,|b|^2\int_0^{\frac{1}{2}T}\int_{T'}^{T-T'} T'^2 A(T-T')\,A(T'')\,A^*(T''-T') \\ &\quad \times \exp\{-\tfrac{1}{3}\lambda\alpha^2 U'^2_c\,T'^2(3T - T' - 3T'')\}\,\mathrm{d}T''\,\mathrm{d}T'. \quad (4.8)\end{aligned}$$

The last term, which is the only nonlinear one, arises from the $\psi_{1x}\psi_{4YYY}$ term in the vorticity equation. The initial condition produces no velocity jump and therefore no jump in $a_{3\pm}(T)$.

5. The evolution equation

Equations (3.4), (3.5,), (3.7), (3.8), (4.2) and (4.5)–(4.8) are now used to determine the jump in $\phi_{3y}\phi-\phi_3\phi'$ in (3.9). This leads to the following evolution equation for A:

$$f_1 A_T+f_2 A+f_3\int_0^{\frac{1}{2}T}\int_{T'}^{T-T'} T'^2 A(T-T')\,A(T'')\,A^*(T''-T') \times\exp\{-\tfrac{1}{3}\lambda\alpha^2 U_c'^2 T'^2(3T-T'-3T'')\}\,\mathrm{d}T''\,\mathrm{d}T' = 0, \quad (5.1)$$

$$f_1 \equiv -\lim_{R\to+\infty}\left[\mathrm{pf}\int_{-R}^{R}\frac{U''-\beta_0}{(U-c)^2}\phi^2\,\mathrm{d}y-\frac{\beta_0\phi_{-\infty}^2\,\mathrm{e}^{-2\mathrm{i}k_-R}}{2\mathrm{i}k_-(U_{-\infty}-c)^2}+\frac{\beta_0\phi_{+\infty}^2\,\mathrm{e}^{2\mathrm{i}k_+R}}{2\mathrm{i}k_+(U_{+\infty}-c)^2}\right] + \mathrm{i}\pi\sum_{y_c}\left\{\frac{-2(U_c''-\beta_0)}{U_c'|U_c'|}a_-b+\frac{(U_c''-\beta_0)U_c''-U_c'U_c'''}{|U_c'|}b^2\right\}, \quad (5.2a)$$

$$f_2 \equiv \alpha\beta_1\left\{-\mathrm{i}\lim_{R\to+\infty}\left[\mathrm{pf}\int_{-R}^{R}\frac{\phi^2}{U-c}\mathrm{d}y+\frac{\phi_{-\infty}^2\,\mathrm{e}^{-2\mathrm{i}k_-R}}{2\mathrm{i}k_-(U_{-\infty}-c)}-\frac{\phi_{+\infty}^2\,\mathrm{e}^{2\mathrm{i}k_+R}}{2\mathrm{i}k_+(U_{+\infty}-c)}\right]+\pi\sum_{y_c}\frac{b^2}{|U_c'|}\right\}, \quad (5.2b)$$

$$f_3 \equiv -\pi\alpha^5\sum_{y_c}U_c'(U_c''-\beta_0)\,b^2\,|b|^2, \quad (5.2c)$$

where pf denotes the finite part of the singular integral as discussed in §4.2, and the constants $a_\pm$ and b are understood to depend on y_c.

The nonlinear term above is a type of integral convolution, and hence is not as simple as those polynomial nonlinearities which often arise in amplitude evolution equations for finite-amplitude waves and instabilities. Its form seems to arise from the fact that the equation for the critical-layer flow is first-order in time and nonhomogeneous. The solution is therefore the convolution of the non-homogeneity and the appropriate Green function. It seems that a similar type of nonlinearity would arise in other finite-amplitude wave and stability problems with the following two characteristics. First, the critical-layer flow is described by a first- or higher-order differential equation in time. That is, the critical-layer flow is not assumed to be steady or quasi-steady. Secondly, the nonlinear interactions inside the critical layers are stronger than those outside. This is even more likely to happen in the case of more-singular modes, for example, those in stratified shear flows.

5.1. *Range of validity*

The above amplitude-evolution equation is only valid for a certain range of times and viscosities. First, time must be large to allow the transients of the initial-value problem to decay, leaving a mode of the Kuo equation. Secondly, time must be much smaller than ν^{-1} so that the effect of the diffusion of the basic flow is not important. Thirdly, the nonlinear terms in the perturbation series for the stream function must not become so large as to destroy the ordering. Specifically, this condition is

$$\epsilon^{\frac{2}{5}}\int_0^{\frac{1}{2}T}\int_{T'}^{T-T'} T'^2 A(T-T')\,A(T'')\,A^*(T''-T') \times\exp\{\tfrac{1}{3}\lambda\alpha^2 U_c'^2\,T'^2(3T-T'-3T'')\}\,\mathrm{d}T''\,\mathrm{d}T' \ll A, \quad (5.3)$$

where the nonlinear term is the size of the velocity jump across the critical layers due to nonlinear interactions inside the critical layers. This nonlinear term also appears in the evolution equation.

The nonlinear term above has the following two bounds:

$$\left| \int_0^{\frac{1}{2}T} \int_{T'}^{T-T'} T'^2 A(T-T')\, A(T'')\, A^*(T''-T') \times \exp\{\tfrac{1}{3}\lambda\alpha^2 U_c'^2\, T'^2(3T-T'-3T'')\}\, \mathrm{d}T''\, \mathrm{d}T' \right| \leqslant \frac{T^4}{96}\|A\|^3, \quad \lambda^{-\frac{4}{3}} Q \|A\|^3,$$

$$\|A\| \equiv \sup_{0 \leqslant T' \leqslant T} |A(T')|,$$

$$Q \equiv (\alpha^2 U_c'^2)^{-\frac{4}{3}} \int_0^{+\infty} \exp\{-\tfrac{2}{3}S^3\}\, \mathrm{d}S.$$

These bounds imply the following sufficient restrictions on viscosity and time:

$$t \ll [\epsilon\|A\|]^{\frac{1}{2}} \quad \text{or} \quad \nu \gg [\epsilon\|A\|]^{\frac{3}{2}}.$$

Thus the evolution equation should be valid at least for the cases of early-enough long times and viscous-enough critical layers.

One might wonder whether the evolution equation can be valid for very long times, even in the absence of viscosity. If the flow is linearly unstable (supercritical), nonlinearity would probably not stabilize it. If, on the other hand, the flow was linearly stable (subcritical), the linear damping term, which is proportional to β_1, could constrain the size of A, and likewise the size of the integral nonlinearity. Disturbances that are initially small enough would probably decay to zero, and the evolution equation in this case would be valid for long times. Disturbances that are initially larger would probably blow up in finite time (subcritical instability). The situation just described is similar to the behaviour of solutions of the evolution equation $A_T = aA + bA|A|^2$ in the case where b is found to be positive. If a is positive then all non-trivial solutions blow up in finite time. If a is negative then solutions blow up if the initial condition is large enough. Only if the initial condition is small enough and the solution decays to zero is the evolution equation valid for long times. There is a non-trivial steady solution for negative a, but the solution is not stable, and therefore not likely to be observed.

If the ordering (5.3) does break down a few notable things happen. Because the effect of nonlinearity has become stronger in the critical layer, a term of the form $\hat{\psi}_{1x}\hat{\psi}_{nYYY}$ that formerly appeared as a non-homogeneity in the equation for $\hat{\psi}_{n+1}$ can no longer be treated as such. The term $\hat{\psi}_{1x}\hat{\psi}_{nYYY}$ now appears in the equation for $\hat{\psi}_n$. In the present formalism the first appearance of such a nonlinearity would be the inclusion of the term $\hat{\psi}_{1x}\hat{\psi}_{3YYY}$ in the equation for $\hat{\psi}_3$. (The quantities $\hat{\psi}_{1YY}$ and $\hat{\psi}_{2YY}$ are identically zero from the requirements that they match to an $O(\epsilon)$ outer solution.) It is the critical layer flow $\hat{\psi}_3$ that determines the velocity jump of the leading order fundamental mode outside the critical layer. If the $\hat{\psi}_{1x}\hat{\psi}_{3YYY}$ term is included in the equation for $\hat{\psi}_3$ it is likely to produce a velocity jump across the critical layer that is composed of harmonics as well as the fundamental. Matching conditions would then require there to be harmonics in the leading-order outer flow ψ_1, and thus also in the leading-order critical-layer flow $\hat{\psi}_1$.

5.2. *Viscous limit*

In the limit of large λ (viscous critical layers) the evolution equation simplifies somewhat. The nonlinear term in (5.1) becomes a cubic, reducing the evolution equation to the following:

$$f_1 \tilde{A}_\tau + \tilde{f}_2 \tilde{A} + \tilde{f}_3 \tilde{A}|A|^2 = 0,$$

$$\tau \equiv \lambda^{-\frac{4}{3}} T, \quad \tilde{A}(\tau) \equiv A(\lambda^{\frac{4}{3}} \tau),$$

$$\tilde{f}_2 = \lambda^{\frac{4}{3}} f_2, \quad \tilde{f}_3 = Q f_3,$$

where Q was defined above. The reason for the simplification of the nonlinearity is that in the case of large λ the $\hat{\psi}_{YYT}$ term in the equations for the critical-layer flow becomes negligible in comparison to the $\lambda \hat{\psi}_{YYYY}$ term. So, the equation is essentially zeroth-order in time, and the flow is quasi-steady. The new time τ is slower than the old time T. The parameter f_2, which corresponds to the departure of β from the critical value, must be smaller than before so that its effect is the same size as that of nonlinearity and time-dependence.

6. Conclusion

There are several contrasts between the present analysis and that of previous workers. These involve the treatment of the critical-layer flow, the magnitude of the velocity jump across a critical layer, the source and form of the nonlinear term in the evolution equation, and the order of the evolution equation.

The critical-layer flow is assumed to be steady or quasisteady by Schade (1964); Benney & Bergeron (1969); Davis (1969); Kelly & Maslowe (1970); Maslowe (1972); Haberman (1972); Benney & Maslowe (1975); Redekopp (1977); Maslowe & Redekopp (1979, 1980); Huerre (1980), Huerre & Scott (1980) and Burns & Maslowe (1983). In the present analysis the critical-layer flow is allowed to be time-dependent. It evolves according to a partial differential equation in time and space. The critical layers are quasi-steady, however, in the viscous limit discussed in §5.2. Thus the assumption of steady or quasi-steady critical layers may be justified in cases where viscosity is the dominant effect in the critical layers. If, on the other hand, the effect of viscosity does not dominate that of nonlinearity, then the present analysis suggests that the critical-layer flow does not become quasi-steady, and should not be assumed to become so. Stewartson (1978) also mentions this point.

The critical layers considered by Benney & Bergeron (1969); Davis (1969); Kelly & Maslowe (1970); Maslowe (1972); Haberman (1972); Benney & Maslowe (1975) and Huerre & Scott (1980) are relatively thin. Also, the nonlinear interactions inside the critical layers are strong, and the equations for the fundamental component and the harmonics are inextricably coupled. Outside the critical layers, however, the equations for the fundamental and the harmonics decouple, and only the fundamental is present at lowest order. In the present analysis the critical layers have not yet become so thin. The weakly nonlinear interactions inside the critical layers, however, have already begun to affect the flow outside. The equations of motion for the fundamental mode and the harmonics decouple everywhere. The ordering studied here may persist for a very long time or may break down eventually as the nonlinearity grows. In the case of breakdown the harmonics are likely to become as large as the fundamental both inside and outside the critical layers as discussed at the end of §5.1. This would suggest that an analysis that assumes strong nonlinear interactions inside the critical layers should include the harmonics as well as the fundamental at lowest order outside the critical layers. Warn & Warn (1978) have done this.

Although Stewartson (1978, 1981); Brown and Stewartson (1978*a*, 1980, 1982*a*, *b*) and Warn & Warn (1978) consider time-dependent critical layers, as is done here, there are significant differences between their analyses and the present one. The former assume stronger nonlinear interactions in the critical layers, but no single nonlinear evolution equation is derived. The present assumes weakly nonlinear critical layers, and it is thereby possible to derive a single evolution equation.

The velocity jump across a nonlinear, inviscid critical layer was found to be zero by Benney & Bergeron (1969); Davis (1969); Kelly & Maslowe (1970); Maslowe (1972); Haberman (1972); Benney & Maslowe (1975); Redekopp (1977); Maslowe & Redekopp (1979, 1980) and Huerre & Scott (1980). There is always a non-zero velocity jump, however, in the present analysis. This is independent of the size of viscosity or boundary conditions. It arises from the time-dependence of the critical-layer flow. The size of the velocity jump, however, corresponds to what is found in critical-layer flows dominated by viscous effects.

The nonlinear terms in the amplitude evolution equations derived by Schade (1964); Benney & Maslowe (1975); Redekopp (1977); Maslowe & Redekopp (1979, 1980) and Burns & Maslowe (1983) are polynomials and come only from nonlinear interactions outside the critical layers. In the present analysis the nonlinear terms in the amplitude evolution equation come only from nonlinear interactions inside the critical layers. This is determined by the scaling argument of §2. The nonlinearity in (5.1) is not a simple polynomial (except in the viscous limit) but is some type of convolution integral. As discussed in §5, this form of the nonlinearity seems to arise because the critical-layer flow is allowed to be time-dependent.

The analysis of Huerre & Scott (1980) resembles the present one in the sense that the nonlinearity in their evolution equation does come from the critical-layer flow and is not simply a polynomial. Their nonlinearity cannot be written in terms of elementary functions, however. The reason is that they considered a more strongly nonlinear critical layer. They also assumed a quasi-steady critical layer rather than a time-dependent one.

The work of Brown & Stewartson (1978*b*) also resembles somewhat the present analysis. The nonlinearity in their evolution equation is not simply a polynomial but $T^6 A\,|A|^4$. Perhaps they would have obtained an integral convolution term, as in (5.1), if they had not in some places treated A as a constant and in other places treated it as a function of time. If A is assumed to be constant and viscosity is zero in (5.1), the nonlinear term there becomes proportional to $T^4 A|A|^2$.

Finally, amplitude-evolution equations for disturbances in marginally stable flows with piecewise-constant basic-state vorticity and density are second-order in time. Such examples are considered by Drazin (1970); Maslowe & Kelly (1970); Weissmann (1979) and Hickernell (1983). In the present analysis, however, the amplitude-evolution equation is first-order in time, even if the flow is marginally stable.

There are several possible extensions of the present results. Wave groups might be studied by including a long spacescale. The perturbation analysis would have to be carried to even higher orders to derive the effects of dispersion in the case of a real group velocity. The methods used here might be applied to problems of marginally stable flows in the beta-plane with regular neutral modes, marginally stable shear flows with density stratification, and problems with disturbances of large horizontal scale (vanishing wavenumber). The evolution equation (5.1) and others derived for disturbances with time-dependent critical layers could be studied numerically and analytically. Previous analyses of finite-amplitude disturbances have often led to the cubic Schrödinger and Korteweg–de Vries equations and their relatives. Solutions of these equations evolve into solitary wavetrains, which have been identified with physically observed phenomena. It would be interesting to know if the solutions of evolution equations for disturbances with time-dependent critical layers decay, equilibrate, become chaotic, or blow up.

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